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# VELOCITY PROFILE AND FRICTION IN A PLANE-PARALLEL CHANNEL WITH A DEVELOPED TURBULENT COMPRESSIBLE GAS FLOW

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**Аннотация**—Основываясь на теории пути перемешивания Прандтля, удалось обобщить обычный логарифмический профиль скоростей на случай течения сжимаемого газа. Обобщенный профиль зависит от ряда параметров, которые определяются путем решения замкнутой системы обыкновенных квазилинейных дифференциальных уравнений первого порядка.

На примере показана деформация профиля скоростей по мере роста числа Маха. Установлено, что при этом коэффициент трения возрастает.

## NOMENCLATURE

$f_\tau$, velocity profile parameter, $f_\tau = -(1/\tau_0)(\partial\tau/\partial y)_1$;
$G$, gas mass flow rate;
$h$, half-width of a plane-parallel channel;
$K$, isentropic exponent;
$\bar{K}$, dimensionless factor, $\bar{K} = (K - 1/2K)$;
$p$, pressure;
$P$, stagnation pressure;
$Pr$, Prandtl number;
$R$, gas constant;
$Re$, Reynolds number;
$u$, longitudinal velocity component;
$v$, transversal velocity component;
$w$, mean mass velocity;
$V$, limit velocity, $V = \sqrt{(R\theta/\bar{K})}$;
$x$, longitudinal coordinate;
$X$, calibrated longitudinal coordinate;
$y$, transversal coordinate.

Greek symbols

$\delta$, viscous sublayer thickness;
$\varepsilon$, turbulent viscosity coefficient;
$\mu$, dynamic viscosity;
$\rho$, density;
$\tau$, shear stress;
$\theta$, stagnation temperature;
$\omega$, degree of velocity profile occupancy, $\omega = (w/u_1)$;
$\zeta$, friction factor,
$\nu$, kinematic viscosity.

Subscripts

$i$, starting channel section;
$\delta$, viscous sublayer boundary;
0, channel wall;
1, channel axis.

CONSIDER a steady-state adiabatic gas flow in a plane-parallel channel $2h$ wide. It is supposed that in the starting section of the channel the flow is developed (i.e. opposite boundary layers have already closed up), the velocity in the centre of the channel in this section is subsonic. Everywhere through the channel

$$Pr = 1, \qquad \left|\frac{\partial p}{\partial x}\right| \gg \left|\frac{\partial p}{\partial y}\right|.$$

Introduce dimensionless values assuming

$$x = \bar{x}h\bar{R}e, \qquad y = \bar{y}h, \qquad u = \bar{u}V, \qquad v = \bar{v}V$$

$$p = pP_{1i}, \qquad \rho = \bar{\rho}\frac{P_{1i}}{R\theta}, \qquad \bar{\theta} = 1$$

$$\mu = \bar{\mu}\,\mu'_{1i}, \qquad \varepsilon = \varepsilon\mu'_{1i}, \qquad \tau = \bar{\tau}\mu'_{1i}(V/h)$$

where $\mu'$ is viscosity-determined at stagnation temperature

$$\bar{R}e = \frac{VhP_{1i}}{\mu'_{1i}R\theta}.$$

On omitting the bars over the dimensionless values, we get the set of boundary-layer equations

$$\rho\left(u\frac{\partial u}{\partial x} + \bar{R}ev\frac{\partial u}{\partial y}\right) = -\bar{K}\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{\partial \tau}{\partial y}, \qquad (1)$$

$$\frac{\partial(\rho u)}{\partial x} + \bar{R}e\frac{\partial(\rho v)}{\partial y} = 0, \qquad (2)$$

$$\tau = (\mu + \varepsilon)\frac{\partial u}{\partial y}, \qquad (3)$$

$$p = \rho(1 - u^2). \qquad (4)$$

From these equations it is not difficult to obtain integral equations

$$\int_0^1 \rho u \,\mathrm{d}y = \tfrac{1}{2}G \qquad (5)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^1 \rho u^2 \,\mathrm{d}y + \bar{K}\frac{\mathrm{d}p}{\mathrm{d}x} + \tau_0 = 0. \qquad (6)$$

Assuming in equation (1) $y = 1$, we get the relation on the channel axis

$$\rho_1 u_1 \frac{\mathrm{d}u_1}{\mathrm{d}x} + \bar{K}p' + f_\tau \tau_0 = 0 \qquad (7)$$

where

$$p' = \frac{\mathrm{d}p}{\mathrm{d}x}, \qquad f_\tau = -\frac{1}{\tau_0}\left(\frac{\partial \tau}{\partial y}\right)_1. \qquad (8)$$

To find the velocity profile, use one of the known expressions for $\tau$. The Prandtl expression is the most convenient

$$\tau = \bar{R}e\rho\left(l\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 \qquad (9)$$

whence for velocities in a turbulent kernel we obtain

$$u = u_\delta + \frac{1}{\sqrt{\bar{R}e}}\int_\delta^y \sqrt{\left(\frac{\tau_0}{\delta}\right)}\frac{\mathrm{d}y}{l}. \qquad (10)$$

Velocity distribution in a viscous sublayer can be found on taking into account that according to equation (1) $(\partial\tau/\partial y)_0 = \bar{K}p'$ and near the wall therefore

$$\tau = \tau_0 + \bar{K}p'y. \qquad (11)$$

To this the following distribution corresponds

$$u = \tau_0 y + \tfrac{1}{2}\bar{K}p'y^2. \qquad (12)$$

Present the value $\sqrt{(\tau/\rho)}$ from equation (11) in the form of

$$\sqrt{(\tau/\rho)} = C_0 + C_1 y + C_2 y^2 + C_3\sqrt{(1 - y)}. \qquad (13)$$

The coefficients $C_i$ can be found from the following conditions

$$\sqrt{\left(\frac{\tau}{\rho}\right)}_{y=0} = C_0 + C_3,$$

$$\sqrt{\left(\frac{\tau}{\rho}\right)}_{y=1} = C_0 + C_1 + C_2.$$

Simple treatments lead to the following expressions*

$$C_0 = \sqrt{\left(\frac{\tau_0}{p}\right)}(1 - \sigma),$$

$$C_1 = \tfrac{1}{2}\sqrt{\left(\frac{\tau_0}{p}\right)}\left(\frac{\bar{K}p'}{\tau_0} + \sigma\right)$$

$$C_2 = \tfrac{1}{2}\sqrt{\left(\frac{\tau_0}{p}\right)}\left(-\frac{\bar{K}p'}{\tau_0} - 2 + \sigma\right),$$

$$C_3 = \sqrt{\left(\frac{\tau_0}{p}\right)}\sigma$$

where

$$\sigma = \sqrt{[f_\tau(1 - u_1^2)]}.$$

* The equality $\tau = (1 - y)f_\tau\tau_0$ is taken into account near $y = 1$.

In case of low velocities when a gas flow slightly differs from an incompressible liquid flow we have $(du_1/dx) \approx 0$,

$$\frac{d}{dx}\int_0^1 \rho u^2 \, dy \approx 0$$

and from equations (7) and (8) obtain

$$-(\bar{K}p'/\tau_0) \approx 1, \qquad f_\tau \approx 1.$$

Besides, $u_1^2 \approx 0$. At low velocities therefore $C_0 \approx C_1 \approx C_2 \approx 0$, $C_3 \approx \sqrt{(\tau_0/p)}$, $\rho \approx p$ and approximation (13) gives

$$\sqrt{(\tau/p)} \approx \sqrt{(\tau_0/p)}\sqrt{(1-y)},$$

i.e. it leads to nearly linear distribution $\tau$ typical for the case of incompressible flow.

By substituting equation (13) into (10) it is not difficult to find the velocity profile in a turbulent kernel if the relation $l$ versus $y$ is known. Let us find such a profile which at small velocities reduces to the known logarithmic one. To this end at sufficiently small $y$ it should be supposed that $\sqrt{(1-y)} = 1$ in equation (13) and $l = \varkappa y = 0{\cdot}4\ y$ in (10) and the obtained profile should be extended to all values of $y$ from $\delta$ to 1.

Substituting equation (13) into (10) and assuming $l = 0{\cdot}4\ y$, we obtain

$$u = u_\delta + \frac{2{\cdot}5}{\sqrt{Re}}\sqrt{(\tau_0/p)}\,[\psi(x, y) - \psi(x, \delta)] \quad (14)$$

where

$$\psi(x, y) = \ln y + a_1 y + \tfrac{1}{2}a_2 y^2;$$

$$a_1 = \tfrac{1}{2}\left(\frac{\bar{K}p'}{\tau_0} + \sigma\right), \quad a_2 = \tfrac{1}{2}\left(-\frac{\bar{K}p'}{\tau_0} - 2 + \sigma\right).$$

For incompressible liquid $a_1 = a_2 = 0$, and the profile (14) will be an ordinary logarithmic one. The profile (14) differs from the incompressible one by the parameter $-(\bar{K}p'/\tau_0)$ and $\sigma$ which describe the effect of velocity and its gradient. Each of the above parameters in the case of incompressible flow is equal to unity.

The right-hand side of equation (14) has 6 unknown values $u_1$, $u_\delta$, $p$, $\tau_0$, $\delta$, $f_\tau$. To determine these unknowns we have 6 equations:

(a) equation (14) written for the axis of the channel

$$u_1 = u_\delta + \frac{2{\cdot}5}{\sqrt{Re}}\sqrt{\left(\frac{\tau_0}{p}\right)}\,[\psi(x, 1) - \psi(x, \delta)];$$

(b) equation of the rate conservation (5);
(c) momentum equation (6);
(d) relation on the axis (7);
(e) condition of the coincidence between the profile (14) with that in a viscous sublayer (12) at $y = \delta$;
(f) condition of viscous sublayer stability [1]

$$\sqrt{Re}\frac{\delta}{\nu_\delta}\sqrt{\left(\frac{\tau_\delta}{\rho_\delta}\right)} = 11{\cdot}6.$$

This set of equations can practically be easily solved when reduced to the set of ordinary quasi-linear differential equations. If in the starting section of the channel the velocity $u_1$ is sufficiently small, one can consider that the developed flow is determined by the relations typical for incompressible liquid flow. This allows the initial conditions to be found.

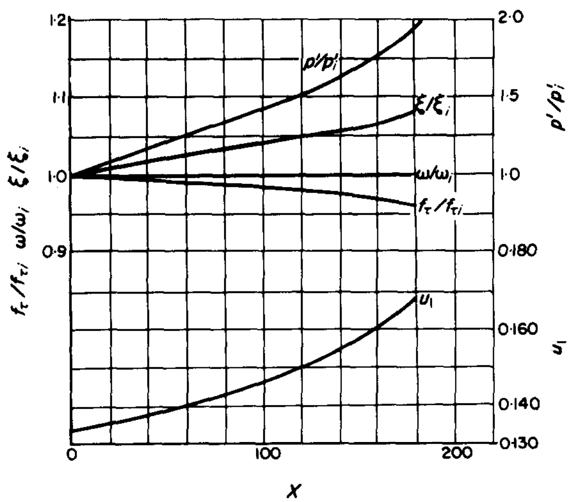


FIG. 1. The velocity on the axis, pressure gradient, degree of profile occupancy, parameter $f_\tau$ and friction factor versus the current length of the channel $X$.

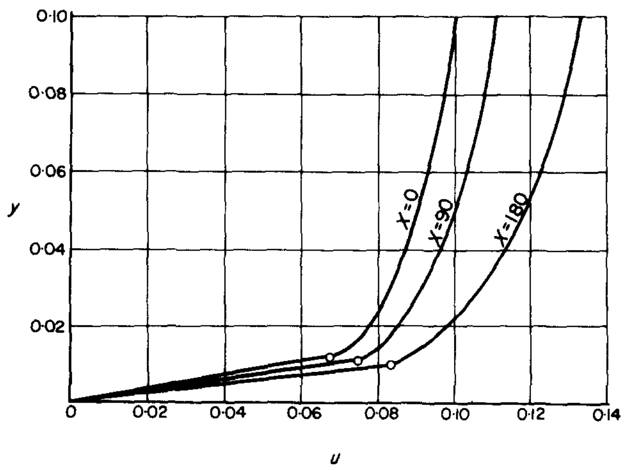


FIG. 2. Velocity profile deformation near the wall. The boundary of the sublayer is shown by circles.

Some calculational results are shown in Figs. 1 and 2. As seen from Fig. 1 the velocity $u_1$ and pressure gradient $|p'|$ increase more intensely with the increase of the current length $X$.

The degree of the profile occupancy $\omega$ does not practically change, and the friction factor $\zeta = (16\tau_0/\bar{R}eGw)$ increases.

The increase of $\zeta$ is determined by essential deformation of the velocity profile near the wall (Fig. 2); the thickness of the viscous sublayer diminishes with simultaneous increase of the velocity on its boundary. This leads to such an increase of $\tau_0$ which effects $\zeta$ more pronouncedly than the increase of the mean mass velocity $w$ does.

Note, that the above method can be applied not only for $\tau$ given in the form of (9). One could, for example, avail oneself of $\tau$ from [2]. It does not however change the main point.

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**Abstract**—On the basis of the Prandtl theory of mixing length the ordinary logarithmic velocity profile was generalized to apply to the case of a compressible gas flow. The generalized profile depends on a number of parameters which are determined by the solution to a closed set of ordinary quasi-linear differential equations of the first order.

The velocity profile deformation with increase of the Mach number is illustrated by way of an example. It is found that in this case the friction factor increases.

**Résumé**—On a généralisé le profil de vitesse logarithmique habituel sur la base de la théorie de la longueur de mélange de Prandtl pour l'appliquer au cas d'un écoulement de gaz compressible. Le profil généralisé dépend de certains paramètres qui sont déterminés par la solution d'un système fermé d'équations différentielles quasi-linéaires du premier ordre.

La déformation du profil de vitesse, lorsque le nombre de Mach augmente est illustrée au moyen d'un exemple. On trouve que dans ce cas le coefficient de frottement augmente.

**Zusammenfassung**—Auf Grund der Prandtl'schen Theorie des Mischungswegs wurde das gewöhnliche logarithmische Geschwindigkeitsprofil verallgemeinert zur Anwendung auf kompressible Gasströmung. Das verallgemeinerte Profil hängt von einer Reihe von Parametern ab, die bestimmt wurden durch die Lösung einer Gruppe gewöhnlicher quasilinearer Differentialgleichungen erster Ordnung.

Die Deformation des Geschwindigkeitsprofils mit zunehmender Machzahl ist mit Hilfe eines Beispiels gezeigt. Dabei wird die Erhöhung des Reibungsfaktors gefunden.